

# MINIMUM $K_{2,3}$ -SATURATED GRAPHS

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**ABSTRACT.** A graph is  $K_{2,3}$ -saturated if it has no subgraph isomorphic to  $K_{2,3}$ , but does contain a  $K_{2,3}$  after the addition of any new edge. We prove that the minimum number of edges in a  $K_{2,3}$ -saturated graph on  $n \geq 5$  vertices is  $\text{sat}(n, K_{2,3}) = 2n - 3$ .

## § 1. Introduction.

All graphs studied are simple ones. We denote a path, a cycle, a star, a complete graph, the complement of a complete graph, and a complete  $r$ -uniform hypergraph with  $n$  vertices by  $P_n$ ,  $C_n$ ,  $S_n$ ,  $K_n$ ,  $I_n$ , and  $K_n^r$ , respectively. We write  $K_{n_1, \dots, n_r}$  for the complete  $r$ -partite graph with partite sets of sizes  $n_1, \dots, n_r$ . For a graph  $G$ , denote  $V = V(G)$ , let  $N(x)$  be the set of the vertices adjacent to  $x$ , and  $d(x) = |N(x)|$ ,  $N[x] = N(x) \cup \{x\}$ ,  $e(G) = |E(G)|$ , and  $\delta(G) = \min\{d(x) : x \in V\}$ . If  $A, B \subseteq V$ , we define  $G[A, B]$  to be the subgraph with vertex set  $A \cup B$  and edge set  $E(G[A, B]) = \{xy \in E(G) : x \in A, y \in B\}$ . We write  $e(G[A, B])$  as  $e([A, B])$ . If  $A = B$ , we write  $G[A, A]$  as  $G[A]$  and  $|E(G[A])|$  as  $e(A)$ . A *clique* in a graph is a set of pairwise adjacent vertices. For two graphs  $G$  and  $H$ , the *disjoint union*  $G + H$  has vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . The *join*  $G * H$  is obtained from  $G + H$  by adding the edges  $\{xy : x \in V(G), y \in V(H)\}$ .

Let  $\mathcal{F}$  be a family of graphs or hypergraphs. A hypergraph is  $\mathcal{F}$ -saturated if it has no  $F \in \mathcal{F}$  as a subhypergraph, but does contain some  $H \in \mathcal{F}$  after the addition of any new edge. The minimum and maximum number of edges in an  $\mathcal{F}$ -saturated graph is denoted by  $\text{sat}(n, \mathcal{F})$  and  $\text{ex}(n, \mathcal{F})$ , respectively. An  $\mathcal{F}$ -saturated graph  $G$  on  $n$  vertices with  $e(G) = \text{sat}(n, \mathcal{F})$  is called a  $\text{sat}(n, \mathcal{F})$ -graph. The problem of determining  $\text{ex}(n, \mathcal{F})$  is Turán's problem. If  $\mathcal{F} = \{F\}$ , we also write  $\text{sat}(n, \mathcal{F})$  as  $\text{sat}(n, F)$ . Erdős, Hajnal, and Moon [9] proved that the  $\text{sat}(n, K_k)$ -graph is  $K_{k-2} * I_{n-k+2}$ . Kászonyi and Tuza [15] determined  $\text{sat}(n, F)$  for  $F = S_k, kK_2, P_k$ , and they proved that  $\text{sat}(n, \mathcal{F}) = O(n)$  for any family  $\mathcal{F}$  of graphs.

As for hypergraphs, Bollobás [4] generalized Erdős, Hajnal, and Moon's results to  $K_k^r$ -saturated hypergraphs. Erdős, Füredi, and Tuza [8] obtained  $\text{sat}(n, F)$  for some particular hypergraphs  $F$  with few edges. Pikhurko [17] proved Tuza's conjecture that  $\text{sat}(n, \mathcal{F}) = O(n^{r-1})$  for all families of  $r$ -uniform hypergraphs whose independence numbers are bounded by a constant. For more results and open problems, see [18].

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For cycles, Ollmann [16] pointed out that the  $\text{sat}(n, C_3)$ -graph is the star  $S_n$ , and he obtained all  $\text{sat}(n, C_4)$ -graphs. Later Tuza [21] gave a shorter proof for  $\text{sat}(n, C_4)$ . Ashkenazi [1] described the properties of  $C_3$ -saturated graphs, planar  $C_3$ -saturated graphs, and  $C_4$ -free  $C_3$ -saturated graphs. Fisher et al. [11] constructed the  $C_5$ -saturated graphs establishing the upper bounds of  $\text{sat}(n, C_5)$ . Chen [6,7] determined all  $\text{sat}(n, C_5)$ -graphs. Barefoot et al. [2] showed that  $n + c_1 n/k \leq \text{sat}(n, C_k) \leq n + c_2 n/k$  for some positive  $c_1, c_2$ . They [2] and Gould, Luczak, and Schmitt [13] gave new upper bounds for  $\text{sat}(n, C_k)$  for small  $k$ . Recently, Füredi and Kim [12] gave almost exact asymptotics for  $\text{sat}(n, C_k)$  as  $k$  is fixed and  $n \rightarrow \infty$ . For more results and open problems, see the excellent survey by Faudree, Faudree, and Schmitt [10].

Pikhurko [17] and G. Chen et al. [5] obtained  $\text{sat}(n, K_{1,\dots,1,\ell})$  of the complete  $(r+1)$ -partite graph  $K_{1,\dots,1,\ell}$  for  $n \geq n(r, \ell)$ . Pikhurko and Schmitt [19] presented  $K_{2,3}$ -saturated graphs with  $2n-3$  edges and proved  $\text{sat}(n, K_{2,3}) \geq 2n - cn^{3/4}$ , where  $c$  is a constant. Gould and Schmitt [14] conjectured that the complete  $r$ -partite graph  $K_{2,\dots,2}$  has  $\text{sat}(n, K_{2,\dots,2}) = \lceil ((4r-5)n - 4r^2 + 6r - 1)/2 \rceil$  and proved it when the minimum degree of the  $K_{2,\dots,2}$ -saturated graphs is  $2r-3$ . Recently, Bohman, Fonoberova, and Pikhurko [3] proved that for  $r \geq 2$  and  $s_r \geq \dots \geq s_1 \geq 1$ , as  $n \rightarrow \infty$ ,  $\text{sat}(n, K_{s_1,\dots,s_r}) = (s_1 + \dots + s_{r-1} + 0.5s_r - 1.5)n + O(n^{3/4})$ . They [3] constructed a  $K_{s_1,\dots,s_r}$ -saturated graph  $K_p * H$  with  $(s_1 + \dots + s_{r-1} + 0.5s_r - 1.5)n + O(1)$  edges, where  $H$  is a  $K_{1,s_r}$ -saturated graph and  $p = s_1 + \dots + s_{r-1} - 1$ . They [3] showed that any  $K_{s_1,\dots,s_r}$ -saturated graph on  $n$  vertices with at most  $\text{sat}(n, K_{s_1,\dots,s_r}) + o(n)$  edges can be transformed into  $K_p * H$  by adding and removing at most  $o(n)$  edges. Bohman, Fonoberova, and Pikhurko [3] also conjectured that  $\text{sat}(n, K_{2,3}) = 2n - 3$ . Here we prove their conjecture.

**Theorem 1.**  $\text{sat}(n, K_{2,3}) = 2n - 3$ .

We present  $\text{sat}(n, K_{2,3})$ -graphs in Section 2, obtain the properties and structures of  $K_{2,3}$ -saturated graphs in Section 3, and prove Theorem 1 in Sections 4-6.

## § 2. Extremal graphs.

In this section, we present  $\text{sat}(n, K_{2,3})$ -graphs. The construction of these graphs was applied to obtain general upper bounds of  $\text{sat}(n, \mathcal{F})$  for any family  $\mathcal{F}$  of graphs by Kászonyi and Tuza [15], and the structure was recently proved to be possessed by all almost extremal  $K_{s_1,\dots,s_r}$ -saturated graphs by Bohman, Fonoberova, and Pikhurko [3]. Pikhurko and Schmitt [19] have presented these  $K_{2,3}$ -saturated graphs with  $2n-3$  edges; we include them for completeness. Let  $\mathcal{H}_n$  be the set of 2-regular  $K_{2,2}$ -free graphs on  $n$  vertices. Let  $\mathcal{G}_n = \{K_1 * (K_1 + H) : H \in \mathcal{H}_{n-2}\} \cup \{K_1 * (K_2 + H) : H \in \mathcal{H}_{n-3}\} \cup \{K_1 * (P_4 + H) : H \in \mathcal{H}_{n-5}\}$ . It is easy to verify that each of the graphs in  $\mathcal{G}_n$  is  $K_{2,3}$ -saturated, since the addition of any edge results in either a  $K_{1,3}$  or a  $C_4$  disjoint from the vertex of maximum degree.

## § 3. Properties of $K_{2,3}$ -saturated graphs.

In this section, let  $G$  be a  $K_{2,3}$ -saturated graph and we shall describe properties of  $G$ .

**Proposition 3.1.** *If  $\alpha_1 \alpha_2 \notin E(G)$ , then there exists a vertex  $b$  in  $N(\alpha_i)$  such that  $b$  and  $\alpha_{3-i}$  have two common neighbors, namely,  $|N(\alpha_{3-i}) \cap N(b)| \geq 2$ ,  $i = 1$  or  $2$ .*

*Proof.* Let  $A, B$  be two partite sets generating a  $K_{2,3}$  obtained by adding  $\alpha_1\alpha_2$  to  $E(G)$ . Let  $A = \{\alpha_i, z, w\}$  and  $B = \{\alpha_{3-i}, b\}$ . Thus  $b \in N(\alpha_i)$  is adjacent to  $z, w \in N(\alpha_{3-i})$ .  $\square$

**Corollary 3.2.** *Let  $N(\alpha) = \{x_1, \dots, x_k\}$ . If  $y \in V \setminus N[\alpha]$ , then either there exists  $l$  such that  $|N(y) \cap N(x_l)| \geq 2$  or there are  $i, j$  such that  $i \neq j$  and  $y$  is adjacent to a vertex in  $N(x_i) \cap N(x_j)$ . In particular, if  $k = 1$ , then for all  $y \in V \setminus N[\alpha]$ , it satisfies  $|N(y) \cap N(x_1)| \geq 2$ .*

**Corollary 3.3.** *Let  $V_1 = N[\alpha] \cup \{v \in V : |N(v) \cap N(\alpha)| = 2\}$ , the set  $\mathcal{U}_2 = \{v \in V \setminus V_1 : |N(v) \cap N(\alpha)| = 1\}$ , and  $\mathcal{U}_3 = V \setminus (V_1 \cup \mathcal{U}_2)$ . We define  $\omega(b) = |N(b) \cap V_1| + 0.5|N(b) \cap \mathcal{U}_2|$  for  $b \in \mathcal{U}_2$ . Let  $x^* \in N(\alpha)$  with  $|N(x^*) \cap N(\alpha)| \leq 1$ . When  $y \in \mathcal{U}_2$  and  $x^*$  is the unique common neighbor of  $\alpha$  and  $y$ , we conclude  $\omega(y) \geq 1.5$  and if  $\omega(y) = 1.5$ , then there exist  $x \in N(\alpha) \cap N(x^*)$  and  $y' \in N(x)$  such that  $N(y) \cap \mathcal{U}_2 = \{y'\}$ . If  $z \in \mathcal{U}_3$ , then we have  $|N(z) \cap (V \setminus \mathcal{U}_3)| \geq 1 + |\{x \in N(\alpha) : N(z) \cap N(x) \cap \mathcal{U}_2 \neq \emptyset\}|$ .*

*Proof.* We assume  $N(y) \cap V_1 = \{x^*\}$ . Since  $|N(x^*) \cap N(\alpha)| \leq 1$ , by Corollary 3.2, there is  $x \in N(\alpha)$  such that  $|N(y) \cap N(x)| \geq 2$ . Thus there is  $y' \in (N(y) \cap N(x)) \setminus \{x^*\}$  and  $\omega(y) \geq 1.5$ . If  $\omega(y) = 1.5$ , then  $N(y) \cap \mathcal{U}_2 = \{y'\}$  and  $N(y) \cap N(x) = \{y', x^*\}$ . Since  $G$  is  $K_{2,3}$ -free, for  $z \in V(G) \setminus \{\alpha\}$ ,  $|N(z) \cap N(\alpha)| \leq 2$ . Let  $z \in \mathcal{U}_3$ . Thus  $z\alpha \notin E(G)$  and  $|N(z) \cap N(\alpha)| = 0$ . Since every vertex in  $N(z) \cap \mathcal{U}_2$  is adjacent to exactly one vertex in  $N(\alpha)$ , it follows  $|N(z) \cap \mathcal{U}_2| \geq |\{x \in N(\alpha) : N(z) \cap N(x) \cap \mathcal{U}_2 \neq \emptyset\}|$ . Let  $|N(z) \cap V_1| = 0$ . Thus every vertex in  $N(z)$  is adjacent to at most one vertex in  $N(\alpha)$ . By Proposition 3.1, there is  $x_1 \in N(\alpha)$  adjacent to two vertices  $w_1, w_2 \in N(z)$ . Since every vertex in  $(N(z) \setminus \{w_1\}) \cap \mathcal{U}_2$  is adjacent to exactly one vertex in  $N(\alpha)$ , the vertex  $w_1$  is adjacent to only  $x_1$  in  $N(\alpha)$ , and  $x_1$  has  $w_2 \in N(x_1) \cap N(z) \cap \mathcal{U}_2$ , it follows  $|N(z) \cap (V \setminus \mathcal{U}_3)| = |\{w_1\}| + |(N(z) \setminus \{w_1\}) \cap \mathcal{U}_2| \geq 1 + |\{x \in N(\alpha) : N(z) \cap N(x) \cap \mathcal{U}_2 \neq \emptyset\}|$ .  $\square$

#### § 4. The case when $\delta(G) = 1$ .

In this section, we prove Theorem 1 when  $\delta(G) = 1$ . Although Pikhurko and Schmitt [19] have proved Theorem 1 when  $\delta(G) = 1$ , for completeness, we include our proof here.

**Lemma 4.1.** *If  $G$  is a  $\text{sat}(n, K_{2,3})$ -graph with  $\delta(G) = 1$ , then  $e(G) \geq 2n - 3$ .*

*Proof.* Let  $N(\alpha) = \{x\}$ ,  $U_1 = N(x) \setminus \{\alpha\}$ , and  $U_2 = V \setminus N[x]$ . By Corollary 3.2, if  $y \in U_1 \cup U_2$ , then  $|N(y) \cap U_1| \geq 2$ . Hence

$$\begin{aligned} e(G) &= d(x) + e(U_1) + e([U_1, U_2]) + e(U_2) = d(x) + 0.5 \sum_{y \in U_1} |N(y) \cap U_1| + \sum_{z \in U_2} |N(z) \cap U_1| + e(U_2) \\ &\geq d(x) + |U_1| + 2|U_2| + e(U_2) = d(x) + (d(x) - 1) + 2(n - d(x) - 1) + e(U_2) \geq 2n - 3 + e(U_2). \quad \square \end{aligned}$$

#### § 5. The case when $\delta(G) = 2$ .

In this section, we prove Theorem 1 when  $\delta(G) = 2$ .

**Lemma 5.1.** *Let  $G$  be a  $\text{sat}(n, K_{2,3})$ -graph with  $\delta(G) = 2$ ,  $A$  be the set of degree 2 vertices having adjacent neighbors, and  $B$  be the set of degree 2 vertices whose neighbors have exactly one common neighbor. If  $A \cap B \neq \emptyset$ , then  $e(G) \geq 2n - 3$ .*

*Proof.* We choose a vertex  $\alpha \in A \cap B$  and denote  $N(\alpha) = \{x_1, x_2\}$ . We define  $V_1 = N[\alpha]$ ,  $\mathcal{U}_2 = (N(x_1) \cup N(x_2)) \setminus V_1$ , and  $\mathcal{U}_3 = V \setminus (V_1 \cup \mathcal{U}_2)$ . For  $y \in \mathcal{U}_2$ , we define

$$\omega(y) = |N(y) \cap V_1| + 0.5|N(y) \cap \mathcal{U}_2| - 2,$$

$$\mathcal{U}_2^+ = \{y \in \mathcal{U}_2 : \omega(y) \geq 0.5\}, \mathcal{U}_2^- = \{y \in \mathcal{U}_2 : \omega(y) < 0\}, \text{ and } \mathcal{U}_2^0 = \mathcal{U}_2 \setminus \mathcal{U}_2^-.$$

For  $z \in \mathcal{U}_3$ , let  $\omega(z) = |N(z) \cap \mathcal{U}_2|$ ,

$$U_3^3 = \{z \in \mathcal{U}_3 : \omega(z) = 3\}, \text{ and } U_3^4 = \{z \in \mathcal{U}_3 : \omega(z) \geq 4\}.$$

For  $y \in \mathcal{U}_2^-$ , we define  $f(y)$ , a subset of  $N(y)$ . We partition  $\mathcal{U}_2^-$  into  $S_0, \dots, S_4$ :

$$S_0 = \{y \in \mathcal{U}_2^- : f(y) \subseteq U_2^+\}, S_4 = \{y \in \mathcal{U}_2^- : f(y) \cap U_3^4 \neq \emptyset\},$$

$$S_1 = \{y \in \mathcal{U}_2^- : |f(y)| = 1, f(y) \subseteq U_3^3\}, S_2 = \{y \in \mathcal{U}_2^- : |f(y)| = 2, f(y) \subseteq U_3^3\}, \text{ and}$$

$$S_3 = \{y \in \mathcal{U}_2^- : f(y) = \{y', z\}, z \in U_3^3, y' \in \mathcal{U}_2\}.$$

Now we define  $f(y)$  and verify that  $\mathcal{U}_2^- = S_0 \cup \dots \cup S_4$ . Let  $y \in \mathcal{U}_2^- \cap N(x_i)$ . By Corollary 3.3,  $\omega(y) = -0.5$  and there is  $y' \in N(x_{3-i}) \setminus V_1$  such that  $N(y) \cap (V_1 \cup \mathcal{U}_2) = \{x_i, y'\}$ . If  $y' \in U_2^+$ , then let  $f(y) = \{y'\}$ . Since  $N(y') \cap N(x_i) = \{x_{3-i}, y\}$ , it follows  $f(b) \neq \{y'\}$  if  $b \neq y$ . Thus

$$|S_0| \leq |U_2^+|. \quad (1)$$

Now we define  $f(y)$  for  $y \in \mathcal{U}_2^-$  whose unique neighbor  $y'$  in  $\mathcal{U}_2$  has  $\omega(y') \leq 0$ . If there is  $z \in N(y)$  adjacent to distinct  $y_1, y_2 \in N(x_{3-i})$ , then let  $f(y) = \{z\}$ . Since  $N(x_1) \cap N(x_2) = \{\alpha\}$ , and  $\omega(y') \leq 0$ , and  $y, y_1, y_2 \in N(z)$ , it follows  $f(y) \subseteq U_3^3 \cup U_3^4$  and  $y \in S_4 \cup S_1$ . Let  $|N(z) \cap N(x_{3-i})| \leq 1$  for all  $z \in N(y)$ . Since  $yx_{3-i} \notin E(G)$ , by Proposition 3.1, there is  $y_3 \in N(x_{3-i})$  adjacent to distinct  $z_1, z_2 \in N(y)$ . We define  $f(y) = \{z_1, z_2\}$ . Since  $N(y) \cap \mathcal{U}_2 = \{y'\}$ , it follows  $\{z_1, z_2\} \not\subseteq \mathcal{U}_2$  and  $f(y) \cap \mathcal{U}_3 \neq \emptyset$ . By Corollary 3.3, if  $z_j \in f(y) \cap \mathcal{U}_3$  and  $|f(y)| = 2$ , then  $z_j \in U_3^3 \cup U_3^4$  and  $y \in S_2 \cup S_3 \cup S_4$ . Recall  $|N(z) \cap \mathcal{U}_2| \geq 4$  for  $z \in U_3^4$ . We observe that

$$(0.5 + 0.5) \sum_{z \in U_3^4} \omega(z) \geq 2|U_3^4| + 0.5|\{y \in \mathcal{U}_2 : N(y) \cap U_3^4 \neq \emptyset\}| \geq 2|U_3^4| + 0.5|S_4|. \quad (2)$$

We shall partition  $U_3^3$  into  $U_{31}^3, U_{32}^3, U_{33}^3$ , and  $U_3^3 \setminus (U_{31}^3 \cup U_{32}^3 \cup U_{33}^3)$  depending on the manner in which  $f(y)$  intersect with  $U_3^3$ . Recall that if  $z \in U_3^3$ , then  $\omega(z) = |N(z) \cap \mathcal{U}_2| = 3$ . If  $y \in N(x_i)$ ,  $f(y) = \{z\}$ , and  $z \in U_3^3$ , then there are  $y_1, y_2$  such that  $N(z) \cap \mathcal{U}_2 \cap N(x_i) = \{y\}$  and  $N(z) \cap \mathcal{U}_2 \cap N(x_{3-i}) = \{y_1, y_2\}$ . Thus if  $z \in U_3^3$ , then  $\{z\}$  is  $f(y)$  for at most one  $y \in S_1$ .

$$\text{Let } U_{31}^3 = \{z \in U_3^3 \cap f(y) : y \in S_1\}. \text{ Thus } |S_1| = |U_{31}^3|. \quad (3)$$

Let  $y_3 \in N(x_i) \cap S_3$ ,  $y' \in \mathcal{U}_2$ ,  $z' \in U_3^3$ , and  $f(y_3) = \{y', z'\}$ . Thus there is  $y_5 \in N(x_{3-i})$  such that  $N(y_3) \cap N(y_5) = \{y', z'\}$ . By Corollary 3.3,  $y' \in N(x_{3-i})$  and  $y_5 \in \mathcal{U}_2^0$ . Since  $z'$

is adjacent to three vertices in  $\mathcal{U}_2$ , one of which is in  $\mathcal{U}_2^- \cap N(x_i)$ , another one of which is in  $\mathcal{U}_2^0 \cap N(x_{3-i})$ , it follows  $z'$  is in  $f(y_3)$  for at most two  $y_3 \in S_3$  and  $z' \notin U_{31}^3$ . We define

$$U_{32}^3 = \{z \in U_3^3 \cap f(y) \cap f(b) : y, b \in S_3, y \neq b\} \text{ and } U_{33}^3 = \{z \in (U_3^3 \cap f(y)) \setminus U_{32}^3 : y \in S_3\}.$$

Thus

$$|S_3| = 2|U_{32}^3| + |U_{33}^3| \quad (4)$$

and  $U_{31}^3, U_{32}^3, U_{33}^3$  are pairwise disjoint. Also if  $z \in U_{32}^3$ , then  $z$  is adjacent to exactly three vertices in  $\mathcal{U}_2$ , one of which is in  $\mathcal{U}_2^0$ , two of which are in  $S_3$ , and  $N(z) \cap S_2 = \emptyset$ . If  $z' \in U_{33}^3$ , then  $z'$  is adjacent to one vertex in  $\mathcal{U}_2^0$ , one vertex in  $S_3$ , and  $|N(z') \cap S_2| \leq 1$ . Since each  $y \in S_2$  is adjacent to two vertices in  $U_3^3$ , and each vertex in  $U_3^3$  is adjacent to exactly three vertices in  $\mathcal{U}_2$ , it follows  $2|S_2| \leq 2|U_{31}^3| + |U_{33}^3| + 3|U_3^3 \setminus (U_{31}^3 \cup U_{32}^3 \cup U_{33}^3)|$  and by (3), (4),

$$2|U_3^3| = |U_{31}^3| + (2|U_{32}^3| + |U_{33}^3|) + |U_{31}^3| + |U_{33}^3| + 2|U_3^3 \setminus (U_{31}^3 \cup U_{32}^3 \cup U_{33}^3)| \geq |S_1| + |S_3| + |S_2|. \quad (5)$$

If  $b \in \mathcal{U}_3$ , then by Corollary 3.3,  $\omega(b) \geq 2$ . By (1), (2), and (5),

$$e(G) = e(V_1) + e([V_1, \mathcal{U}_2]) + e(\mathcal{U}_2) + e([\mathcal{U}_2, \mathcal{U}_3]) + e(\mathcal{U}_3)$$

$$\geq 2|V_1| - 3 + 2|\mathcal{U}_2| + 0.5(|U_2^+| - |S_0| - \dots - |S_4|) + 2|\mathcal{U}_3| + |U_3^3| + 0.5|S_4| + e(\mathcal{U}_3) \geq 2n - 3. \quad \square$$

**Lemma 5.2.** *Let  $G$  be a  $\text{sat}(n, K_{2,3})$ -graph with  $\delta(G) = 2$  and  $A$  be the set of degree 2 vertices having adjacent neighbors. If  $A \neq \emptyset$ , then  $e(G) \geq 2n - 3$ .*

*Proof.* By Lemma 5.1, since  $G$  is  $K_{2,3}$ -free, we can assume that for each  $\alpha \in A$ , there is a unique vertex  $b$  such that  $N(\alpha) \subseteq N(b)$  and  $b \neq \alpha$ . Let  $\mathcal{B} = \{b \in V : N(b) \supseteq N(a) \text{ for a vertex } a \in A \text{ with } a \neq b\}$ . We choose  $\alpha \in A$  satisfying the unique vertex  $\beta$  with  $N(\alpha) \subseteq N(\beta)$  and  $\beta \neq \alpha$  has  $d(\beta) = \min\{d(b) : b \in \mathcal{B}\}$ . Let  $N(\alpha) = \{x_1, x_2\}$ . Let  $V_1 = \mathcal{U}_1 = N[\alpha] \cup \{\beta\}$ ,  $\mathcal{U}_2 = (N(x_1) \cup N(x_2)) \setminus V_1$ , the set  $\mathcal{U}_3 = V \setminus (V_1 \cup \mathcal{U}_2 \cup N(\beta))$ , and  $\mathcal{U}_4 = V \setminus (V_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3)$ .

$$\text{For } i \in \{2, 3, 4\} \text{ and } y \in \mathcal{U}_i, \text{ let } \omega(y) = |N(y) \cap (\mathcal{U}_1 \cup \dots \cup \mathcal{U}_{i-1})| + 0.5|N(y) \cap \mathcal{U}_i| - 2. \quad (6)$$

$$\text{Let } U_2^+ = \{y \in \mathcal{U}_2 : \omega(y) \geq 0.5\}, \text{ the set } \mathcal{U}_2^- = \{y \in \mathcal{U}_2 : \omega(y) < 0\}, \text{ and } \mathcal{U}_2^0 = \mathcal{U}_2 \setminus \mathcal{U}_2^-.$$

$$\text{Let } U_3^0 = \{z \in \mathcal{U}_3 : \omega(z) = 0.5\}, U_3^1 = \{z \in \mathcal{U}_3 : 1 \leq \omega(z) < 2\}, \text{ and } U_3^2 = \{z \in \mathcal{U}_3 : \omega(z) \geq 2\}.$$

Since  $\alpha\beta \notin E(G)$ , by Proposition 3.1, without loss of generality, there is  $\gamma \in N(x_1) \cap N(\beta)$ . For  $y \in \mathcal{U}_2^-$ , we define  $f(y)$ , a subset of  $N(y)$ . For one particular case we also define  $f(\gamma)$  and leave  $f(y)$  undefined for exactly one  $y \in \mathcal{U}_2^-$ . We partition  $\mathcal{U}_2^- \cup \{\gamma\}$  into  $S_0, \dots, S_6$ :

$$\text{let } S_0 = \{y \in \mathcal{U}_2^- : f(y) \subseteq U_2^+, f(y) \neq \emptyset\}, S_i = \{y \in \mathcal{U}_2^- \cup \{\gamma\} : |f(y)| = i, f(y) \subseteq U_3^1\}, i = 1, 2,$$

$$S_4 = \{y \in \mathcal{U}_2^- \cup \{\gamma\} : f(y) \cap U_3^2 \neq \emptyset\}, S_5 = \{y \in \mathcal{U}_2^- \cup \{\gamma\} : f(y) \cap \mathcal{U}_4 \neq \emptyset, f(y) \cap U_3^2 = \emptyset\},$$

$$S_3 = \{y \in \mathcal{U}_2^- : f(y) = \emptyset\}, \text{ and } S_6 = \{y \in \mathcal{U}_2^- : f(y) = \{z\}, z \in U_3^0\}.$$

Note that  $S_0, \dots, S_6$  are pairwise disjoint. Now we define  $f(y)$ , verify that each  $f(y)$  is as described in the definitions of  $S_0, \dots, S_6$ ,  $|\mathcal{U}_2^-| = |S_0| + \dots + |S_6|$ , and  $f(y)$  satisfies that

$$d(w) \geq 3 \text{ for } w \in \mathcal{U}_4 \cap f(y), \text{ and if } y \in S_1 \text{ and } z \in f(y), \text{ then } N(z) \cap ((\mathcal{U}_2^0 \setminus \{\gamma\}) \cup \mathcal{U}_3) \neq \emptyset. \quad (7)$$

Let  $y \in \mathcal{U}_2^- \cap N(x_i)$ . By Corollary 3.3,  $\omega(y) = -0.5$  and there is  $y' \in N(x_{3-i}) \setminus V_1$  such that  $N(y) \cap (V_1 \cup \mathcal{U}_2) = \{x_i, y'\}$ . If  $y' \in U_2^+$ , then we define  $f(y) = \{y'\}$  and  $y \in S_0$ . Since  $N(y') \cap N(x_i) = \{x_{3-i}, y\}$ , it follows  $f(b) \neq \{y'\}$  if  $b \neq y$ . Since  $N(\gamma) \cap N(x_2) = \{x_1, \beta\}$ , it follows  $\{\gamma\} \neq f(y)$  for any  $y \in S_0$ . After defining  $f(y)$  for all  $y \in \mathcal{U}_2^-$  having a neighbor in  $U_2^+$ , we conclude

$$|S_0| \leq |U_2^+ \setminus \{\gamma\}|. \quad (8)$$

Let  $y \in \mathcal{U}_2^- \setminus S_0$ . If there is  $z \in U_3^2 \cap N(y)$  or  $z \in \mathcal{U}_4 \cap N(y)$  with  $d(z) \geq 3$ , then we choose one such  $z$  and define  $f(y) = \{z\}$ . We have  $y \in S_4 \cup S_5$  and  $f(y)$  satisfies (7).

Now we define  $f(y)$  for  $y \in \mathcal{U}_2^- \setminus (S_0 \cup S_4 \cup S_5)$ . First let there be  $y_1, y_2 \in \mathcal{U}_2^- \cap N(x_i)$  whose  $f(y_1), f(y_2)$  are to be defined. By Corollary 3.3, there is  $y_{j+2} \in N(x_{3-i})$  such that  $N(y_j) \cap (V_1 \cup \mathcal{U}_2) = \{x_i, y_{j+2}\}$ ,  $j = 1, 2$ . Since  $y_1 y_2 \notin E(G)$ , by Proposition 3.1, without loss of generality, there is  $b \in N(y_1)$  adjacent to  $z_1, z_2 \in N(y_2)$ . Since  $\mathcal{U}_4 \subset N(\beta)$ ,  $y_1, y_2 \notin S_4 \cup S_5$ , and  $y_2 \in \mathcal{U}_2^-$ , it follows  $b \in \{y_3\} \cup \mathcal{U}_3$  and  $z_1, z_2 \in \{y_4\} \cup \mathcal{U}_3$ . Let  $b = y_3$ . We define  $f(y_2) = \{z_1, z_2\} \setminus \{y_4\}$ . By Corollary 3.3, since  $y_2 \notin S_4$ , it follows  $f(y_2) \subseteq U_3^1$  and  $y_2 \in S_1 \cup S_2$ . If  $f(y_2) = \{z_j\}$ , then  $y_4, y_1 \in N(b) \cap \mathcal{U}_2$ , the vertex  $b \in (\mathcal{U}_2^0 \setminus \{\gamma\}) \cap N(z_j)$ , and  $f(y_2)$  satisfies (7). Let  $b \in \mathcal{U}_3$ . We define  $f(y_1) = \{b\}$ . By Corollary 3.3,  $b \in U_3^1$  and  $y_1 \in S_1$ . Since  $\{z_1, z_2\} \cap \mathcal{U}_3 \cap N(b) \neq \emptyset$ , it follows  $f(y_1)$  satisfies (7). We have defined one of  $f(y_1), f(y_2)$ . We repeat this process until there is at most one  $y \in \mathcal{U}_2^- \cap N(x_i)$  whose  $f(y)$  is not defined.

If there is only one  $y \in \mathcal{U}_2^-$  whose  $f(y)$  has not been defined, then we define  $f(y) = \emptyset$  and  $S_3 = \{y\}$ . Let  $u_i^*$  be the unique vertex in  $\mathcal{U}_2^- \cap N(x_i)$  whose  $f(u_i^*)$  is not defined,  $i = 1, 2$ . Recall  $\gamma \in N(\beta) \cap N(x_1)$ . If  $\gamma \in U_2^+$ , then let  $f(u_1^*) = f(u_2^*) = \emptyset$ . Thus  $S_3 = \{u_1, u_2\}$ . By (8),

$$|U_2^+| - |S_0| - |S_3| \geq -1. \quad (9)$$

Let  $\omega(\gamma) = 0$  and  $N(\gamma) \cap (V_1 \cup \mathcal{U}_2) = \{x_1, \beta\}$ . Let there be  $w \in N(u_1^*) \cap \mathcal{U}_4$ . Since  $f(u_1^*)$  is not defined,  $N(w) = \{u_1^*, \beta\}$ . Since  $N(u_1^*) \cap N(\beta) = \{x_1, w\}$  and  $u_2^* w \notin E(G)$ , by Proposition 3.1, there is  $b \in N(w)$  adjacent to  $w_1, w_2 \in N(u_2^*)$ . If  $b = \beta$ , then  $w_1$  or  $w_2$ , say  $w_1 \in N(\beta) \setminus \{x_2\}$ . Since  $u_2^* \in \mathcal{U}_2^-$  and  $f(u_2^*)$  is not defined, it follows  $w_1 \notin \mathcal{U}_2$ ,  $w_1 \in \mathcal{U}_4$ , and  $N(w_1) = \{\beta, u_2^*\}$ , a contradiction to Proposition 3.1, since  $ww_1 \notin E(G)$ . Hence  $b = u_1^*$ . Since  $\mathcal{U}_4 \subset N(\beta)$ , by Corollary 3.3,  $w_1, w_2 \in U_3^1$ . We define  $f(u_1^*) = f(u_2^*) = \{w_1, w_2\}$  and  $u_1^*, u_2^* \in S_2$ .

Let  $N(u_1^*) \cap \mathcal{U}_4 = \emptyset$ . We define  $f(u_2^*) = \emptyset$  and  $u_2^* \in S_3$ . We consider the  $K_{2,3}$  created by adding  $u_1^* \gamma$ . Let there be  $z \in N(u_1^*)$  adjacent to  $w_1, w_2 \in N(\gamma)$ . Let  $z \in \mathcal{U}_2$ . By Corollary 3.3,  $z \in N(x_2)$  and  $w_1, w_2 \in U_3^1 \cup U_3^2 \cup \mathcal{U}_4$ . We define  $f(\gamma) = \{w_1, w_2\}$ , leave  $f(u_1^*)$  undefined, and  $\gamma \in S_2 \cup S_4 \cup S_5$ . Note that  $|\mathcal{U}_2^-| = |S_0| + \dots + |S_6|$ . Since  $\mathcal{U}_4 \subset N(\beta)$ ,

$$\text{if } w \in \mathcal{U}_4 \cap f(\gamma), \text{ then } |N(w) \cap (V \setminus \mathcal{U}_4)| \geq 3 \text{ and } f(\gamma) \text{ satisfies (7)}. \quad (10)$$

Next let  $z \in \mathcal{U}_3$ . We define  $f(u_1^*) = \{z\}$ . Since  $x_1 \in N(\gamma) \cap N(\beta)$ , it follows  $\{w_1, w_2\} \cap \mathcal{U}_3 \cap N(z) \neq \emptyset$ . By Corollary 3.3,  $u_1^* \in S_6 \cup S_1$  and  $f(u_1^*)$  satisfies (7). Next let  $|N(z) \cap N(\gamma)| \leq 1$  for all  $z \in N(u_1^*)$ . By Proposition 3.1, there is  $z_1 \in N(\gamma)$  adjacent to  $z_2, z_3 \in N(u_1^*)$ . Since  $N(u_1^*) \cap \mathcal{U}_4 = \emptyset$  and  $N(\gamma) \cap (V_1 \cup \mathcal{U}_2) = \{x_1, \beta\}$ , it follows  $z_1 \in \mathcal{U}_3 \cup \mathcal{U}_4$ . Let  $z_1 \in \mathcal{U}_3$ . We choose  $z_j \in \{z_2, z_3\} \setminus \mathcal{U}_2$  and define  $f(u_1^*) = \{z_j\}$ . By Corollary 3.3,  $u_1^* \in S_1 \cup S_6$ . Since  $z_1 \in \mathcal{U}_3 \cap N(z_j)$ ,  $f(u_1^*)$  satisfies (7). If  $z_1 \in \mathcal{U}_4$ , then let  $f(u_1^*) = \emptyset$ . Since  $z_1 \in N(\beta)$ , by (8),

$$|U_2^+| - |S_0| - |S_3| \geq -2, \text{ and if (9) fails, then there is } z_1 \in \mathcal{U}_4 \text{ with } |N(z_1) \cap (V \setminus \mathcal{U}_4)| \geq 4. \quad (11)$$

Next we compare  $\sum \omega(z)$  for  $z \in U_3^2$  with  $|S_4|$ . Denote  $\phi(z) = |N(z) \cap \mathcal{U}_2| + 0.5|N(z) \cap \mathcal{U}_3|$ . If  $z \in U_3^2$ , then  $\phi(z) \geq 4$  and  $\omega(z) = \phi(z) - 2 \geq \phi(z) - 0.5\phi(z) \geq 0.5|N(z) \cap \mathcal{U}_2|$ . Hence

$$2 \sum \{\omega(z) : z \in U_3^2\} \geq \sum \{|N(z) \cap \mathcal{U}_2| : z \in U_3^2\} \geq |\{y \in \mathcal{U}_2 : N(y) \cap U_3^2 \neq \emptyset\}| \geq |S_4|. \quad (12)$$

Recall  $3 \leq |N(z) \cap \mathcal{U}_2| + 0.5|N(z) \cap \mathcal{U}_3| \leq 3.5$  for  $z \in U_3^1$ . Since the vertices in  $S_1$  satisfy (7), if  $z \in U_3^1$  and  $|\{b \in S_1 : z \in f(b)\}| = 3$ , then  $\omega(z) = 1.5$ . We partition  $U_3^1$  into six sets:

Let  $U_{3i}^1 = \{z \in U_3^1 : |\{b \in S_1 : z \in f(b)\}| = i, \omega(z) = 1.5\}$ ,  $i = 1, 2, 3$ , the set  $U_{3k}^1 = \{z \in U_3^1 : |\{b \in S_1 : z \in f(b)\}| = k - 3, \omega(z) = 1\}$ ,  $k = 4, 5$ , and  $U_{36}^1 = U_3^1 \setminus (U_{31}^1 \cup \dots \cup U_{35}^1)$ . Thus

$$|S_1| = |U_{31}^1| + |U_{34}^1| + 2(|U_{32}^1| + |U_{35}^1|) + 3|U_{33}^1|. \quad (13)$$

Since each  $y \in S_2$  is adjacent to two vertices in  $f(y) \cap U_3^1$ , and the vertices in  $S_1$  satisfy (7), it follows  $2|S_2| \leq 2|U_{31}^1| + |U_{32}^1| + |U_{34}^1| + 3|U_{36}^1|$ , and by (13),

$$\sum \{\omega(z) : z \in U_3^1\} \geq 1.5(|U_{31}^1| + |U_{32}^1| + |U_{33}^1|) + |U_{34}^1| + |U_{35}^1| + |U_{36}^1| \geq 0.5(|S_1| + |S_2|). \quad (14)$$

Let  $w \in \mathcal{U}_4$ . Since  $w \in N(\beta)$ , the vertex  $\beta \in N(x_k) \cap N(w)$ , and  $G$  is  $K_{2,3}$ -free, it follows  $|N(w) \cap \mathcal{U}_2 \cap N(x_k)| \leq 1$ ,  $k = 1, 2$ , and  $|N(w) \cap S_5| \leq 2$ . We partition  $\mathcal{U}_4$  into three sets. Let  $U_{42} = \{w \in \mathcal{U}_4 \cap f(y) \cap f(b) : y, b \in S_5, y \neq b\}$ ,  $U_{41} = \{w \in (\mathcal{U}_4 \cap f(y)) \setminus U_{42} : y \in S_5\}$ , and  $U_4^* = \mathcal{U}_4 \setminus (U_{41} \cup U_{42})$ . Thus  $|S_5| \leq 2|U_{42}| + |U_{41}|$ . If  $b \in \mathcal{U}_3$ , then by Corollary 3.3,  $\omega(b) \geq 0$ . Since  $f(\gamma)$  is defined when  $u_1^*$  is the only vertex in  $\mathcal{U}_2^-$  whose  $f(u_1^*)$  is undefined, it follows  $|\mathcal{U}_2^-| = |S_0| + \dots + |S_6|$ . Note that  $S_6 \subseteq \{u_1^*\}$ ,  $|S_6| \leq 1$ , and  $|S_6| \leq |U_3^0|$ . By (6), (12), and (14),

$$\begin{aligned} e(G) &= e(V_1) + e([V_1, \mathcal{U}_2]) + e(\mathcal{U}_2) + e([\mathcal{U}_2, \mathcal{U}_3]) + e(\mathcal{U}_3) + e([V \setminus \mathcal{U}_4, \mathcal{U}_4]) + e(\mathcal{U}_4) \\ &\geq 2|V_1| - 3 + 2|\mathcal{U}_2| + 0.5(|U_2^+| - \sum_{0 \leq i \leq 6} |S_i|) + 2|\mathcal{U}_3| + \sum_{z \in U_3^1 \cup U_3^0} \omega(z) + 0.5|S_4| + 2|\mathcal{U}_4| + \sum_{w \in \mathcal{U}_4} \omega(w) \\ &\geq 2n - 3 + 0.5(|U_2^+| - |S_0| - |S_3|) + \sum_{w \in U_{42}} (\omega(w) - 1) + \sum_{w \in U_{41}} (\omega(w) - 0.5) + \sum_{w \in U_4^*} \omega(w). \end{aligned} \quad (15)$$

If  $w \in U_{42}$ , then by Corollary 3.3,  $w$  is adjacent to three vertices in  $V \setminus \mathcal{U}_4$  and  $\omega(w) \geq 1$ . If  $w \in U_{41}$ , then  $|N(w) \cap (V \setminus \mathcal{U}_4)| \geq 2$ , and since  $f(y)$  satisfies (7) for  $y \in \mathcal{U}_2^- \cup \{\gamma\}$ , it follows  $d(w) \geq 3$  and  $\omega(w) \geq 0.5$ . Let  $W = \{w \in U_4^* : \omega(w) < 0\}$  and  $w \in W$  with  $N(w) = \{\beta, u\}$ . We define  $p(w) = u$ , the neighbor of  $w$  in  $\mathcal{U}_4$ . By our assumption, there is a unique vertex  $z$  such that  $N(\beta) \cap N(u) = \{w, z\}$ . We define  $h(w) = z$ . We define

$$W_1 = \{w \in W : h(w) \in V \setminus \mathcal{U}_4\}, \text{ the set } P = \{p(w) : w \in W_1\},$$

$$W_2 = W \setminus W_1 = \{w \in W : h(w) \in \mathcal{U}_4\}, \text{ and } H = \{h(w) : w \in W_2\}.$$

Let  $u \in P$ . Thus there is  $w \in W_1$  such that  $u = p(w)$ . Since  $N(u) \cap N(\beta) = \{w, h(w)\}$  and  $h(w) \in \mathcal{U}_2$ , it follows  $\omega(u) \geq 0.5$ , the vertex  $h(w)$  is not adjacent to any vertex in  $W$ ,  $u \notin W \cup H$ , and  $u \neq p(w')$  for any  $w' \in W \setminus \{w\}$ . Thus  $P \cap (W \cup H) = \emptyset$  and  $|W_1| = |P|$ . Let  $u \in P \cap (U_{42} \cup U_{41})$ . Thus there is  $y \in \mathcal{U}_2^- \cup \{\gamma\}$  such that  $u \in f(y)$ . If  $y = \gamma$ , then by (10),  $|N(u) \cap (V \setminus \mathcal{U}_4)| \geq 3$ . If  $y \in \mathcal{U}_2^-$ , then  $y \notin N(\beta)$  and  $y, h(w), \beta \in N(u) \cap (V \setminus \mathcal{U}_4)$ . Since  $w \in N(u) \cap \mathcal{U}_4$ , it follows  $\omega(u) - 1 \geq 0.5$ . Let  $z \in H$ . Thus there is  $w \in W_2$  such that

$z = h(w)$  and  $p(w) \in N(z) \cap N(\beta)$ . Since  $|N(z) \cap N(\beta)| \leq 2$ , it follows  $z = h(w)$  for at most two  $w$  in  $W_2$ . Thus  $|W_2| \leq 2|H|$ . Since  $w \in A$ , by our choice of  $\alpha$ ,  $d(z) \geq d(\beta) \geq 5$ ,  $\omega(z) \geq 1$  and  $z \notin W$ . If  $z \in U_{41}$ , then  $z$  is adjacent to two vertices in  $V \setminus \mathcal{U}_4$ , and  $\omega(z) - 0.5 \geq 1$ . If  $z \in U_{42}$ , then  $z$  is adjacent to three vertices in  $V \setminus \mathcal{U}_4$ , and  $\omega(z) - 1 \geq 1$ . If (9) holds, then by (15),  $e(G) \geq 2n - 3.5 - 0.5|W_1| - 0.5|W_2| + 0.5|P| + |H| \geq 2n - 3.5$ .

Suppose (9) does not hold. By (11), there is  $z_1 \in \mathcal{U}_4$  with  $|N(z_1) \cap (V \setminus \mathcal{U}_4)| \geq 4$  and  $\omega(z_1) - 1 \geq 1$ . Recall  $P \cap H = \emptyset$ . If  $z_1 \in H$ , then since  $d(z_1) \geq 5$ , we have  $\omega(z_1) - 1 \geq 1.5$ . By (11) and (15),  $e(G) \geq 2n - 4 - 0.5|W| + 0.5|P \setminus \{z_1\}| + |H \setminus \{z_1\}| + (\omega(z_1) - 1) \geq 2n - 3.5$ .  $\square$

**Lemma 5.3.** *If  $G$  is a  $\text{sat}(n, K_{2,3})$ -graph with  $\delta(G) = 2$ , then  $e(G) \geq 2n - 3$ .*

*Proof.* By Lemma 5.2, we assume that degree 2 vertices have nonadjacent neighbors. Let  $B$  be the set of degree 2 vertices whose neighbors have exactly one common neighbor. Let  $\alpha \in B$ , if  $B \neq \emptyset$ . Let  $N(\alpha) = \{x_1, x_2\}$ . If  $|B| = 0$ , then  $N(x_1) \cap N(x_2) = \{\alpha, \beta\}$ . If  $\alpha \in B$ , then we define  $V_1 = N[\alpha]$ , the set  $U_2 = (N(x_1) \cup N(x_2)) \setminus V_1$ , and  $U_3 = V \setminus (V_1 \cup U_2)$ , otherwise, let  $V_1 = N[\alpha] \cup \{\beta\}$ ,  $U_2 = (N(x_1) \cup N(x_2)) \setminus V_1$ , the set  $U_3 = V \setminus (V_1 \cup U_2 \cup N(\beta))$ , and  $U_4 = V \setminus (V_1 \cup U_2 \cup U_3)$ . For  $w \in U_4$ , let  $\omega(w) = |N(w) \cap (V \setminus U_4)| + 0.5|N(w) \cap U_4| - 2$ ,

$$\ell = \sum_{w \in U_4} \omega(w), \quad \theta_2 = \sum_{y \in U_2} (|N(y) \cap V_1| + 0.5|N(y) \cap U_2| - 2), \quad \text{and} \quad \theta_3 = \sum_{z \in U_3} (|N(z) \cap U_2| - 2).$$

Hence

$$\begin{aligned} e(G) &= e(V_1) + e([V_1, U_2]) + e(U_2) + e([U_2, U_3]) + e(U_3) + e([V \setminus U_4, U_4]) + e(U_4) \\ &= 2|V_1| - 4 + 2|U_2| + \theta_2 + 2|U_3| + \theta_3 + e(U_3) + 2|U_4| + \ell = 2n - 4 + \theta_2 + \theta_3 + e(U_3) + \ell. \end{aligned}$$

We shall prove  $\theta_2 + \theta_3 + e(U_3) + \ell \geq 0.5$ . By Corollary 3.3,  $\theta_2, \theta_3 \geq 0$ . First let  $\alpha \in B$ . Thus  $N(x_1) \cap N(x_2) = \{\alpha\}$  and  $|U_4| = \ell = 0$ . Since  $x_1 x_2 \notin E(G)$ , by Proposition 3.1, without loss of generality, there is  $z \in N(x_2)$  adjacent to  $y_1, y_2 \in N(x_1)$ . If  $|N(y_i) \cap U_2| \geq 3$ , then  $\theta_2 \geq 0.5$ . By Corollary 3.2,  $|N(y) \cap N(x_i)| \geq 2$  for  $y \in U_2$ ,  $i = 1$  or  $2$ . Thus let  $N(y_i) \cap U_2 = \{z, z_i\}$ , where  $z_i \in N(x_2)$ ,  $i = 1, 2$ . Since  $y_1 y_2 \notin E(G)$ , by Proposition 3.1, there is  $b \in N(y_k) \setminus \{x_1\}$  adjacent to  $b_1, b_2 \in N(y_{3-k})$ ,  $k = 1$  or  $2$ . Since  $d(b) \geq 3$ , if  $b \in U_3$ , then  $\theta_3 + e(U_3) \geq 1$ . Thus  $b \in \{z, z_k\}$ . Since  $y_{3-k} \in N(x_1) \cap N(b_i)$  and  $b \in N(x_2) \cap N(b_i)$ , by Corollary 3.3, if  $b_i \in U_3$ , then  $\theta_3 \geq 1$ . Thus  $\{b_1, b_2\} = \{z, z_{3-k}\}$ . Hence  $b = z_k$ , and  $y_k, z, z_{3-k} \in N(z_k)$ , and  $\theta_2 \geq 0.5$ .

Finally, let  $B = \emptyset$ . Since degree 2 vertices have nonadjacent neighbors, if  $w \in U_4$  has  $N(w) = \{\beta, u\}$ , then  $u \in V \setminus U_4$  and  $\omega(w) \geq 0$ . Hence  $\ell \geq 0$ . Since  $\beta \notin N(\alpha)$ , by Corollary 3.2,  $x_i$  is adjacent to  $b_1, b_2$  in  $N(\beta)$ ,  $i = 1$  or  $2$ . If  $b_1 \in N(b_2)$ , then  $\theta_2 \geq 1$ . Let  $b_1 \notin N(b_2)$ . By Proposition 3.1, there is  $y \in N(b_j)$  adjacent to  $y_1, y_2 \in N(b_{3-j})$ ,  $j = 1$  or  $2$ . If  $z \in \{y, y_1, y_2\}$  is in  $U_4 \cap N(b_k)$ , then since degree 2 vertices have nonadjacent neighbors,  $N(z) \supsetneq \{\beta, b_k\}$  and  $\ell \geq 0.5$ . Let  $\{y, y_1, y_2\} \cap U_4 = \emptyset$ . If one of  $y_1, y_2, y$  is in  $U_2 \cup \{x_i\}$ , then  $\theta_2 \geq 0.5$ . Hence  $y, y_1, y_2 \in U_3$  and  $e(U_3) \geq 2$ .  $\square$

## § 6. The case when $\delta(G) = 3$ .

In this section, we prove Theorem 1 when  $\delta(G) = 3$ . We define  $\lambda(G) = \min\{e(N(\alpha)) : d(\alpha) = 3\}$  and discuss  $e(G)$  in cases depending on  $\lambda(G)$ .



**Lemma 6.1.** *If  $G$  is a  $\text{sat}(n, K_{2,3})$ -graph with  $\delta(G) = 3$  and  $\lambda(G) = 3$ , then  $e(G) \geq 2n - 2$ .*

*Proof.* Let  $R = \{v \in V : d(v) = 3\}$ . We claim that there is  $\beta \in V$  such that  $R \subseteq N(\beta)$ . Let  $\alpha_1 \in R$  and  $\alpha_2 \in R \setminus N[\alpha_1]$ . If  $N(\alpha_1) = \{b_1, b_2, b_3\}$ , then since  $G$  is  $K_{2,3}$ -free, it follows  $N(b_1) \cap N(b_2) = \{\alpha_1, b_3\}$ . Thus  $|N(\alpha_1) \cap N(\alpha_2)| \leq 1$ . We first show  $|N(\alpha_1) \cap N(\alpha_2)| = 1$ . By Proposition 3.1, there exists  $\beta \in N(\alpha_{3-i})$  adjacent to  $y_1, y_2 \in N(\alpha_i)$ ,  $i = 1$  or  $2$ . We denote  $N(\alpha_i) = \{y, y_1, y_2\}$ . Since  $G$  is  $K_{2,3}$ -free and  $N(\alpha_i)$  is a clique, it follows  $N(y_1) \cap N(y_2) = \{\alpha_i, y\}$ . Thus  $\beta = y$  and  $\{\beta\} = N(\alpha_1) \cap N(\alpha_2)$ . Let  $\alpha_3 \in R \setminus N(\beta)$ . Thus  $\alpha_3 \notin N(\alpha_i)$  and  $N(\alpha_3) \cap (N(\alpha_i) \setminus N(\alpha_{3-i})) = \{z_i\}$ ,  $i = 1, 2$ . We denote  $N(\alpha_3) = \{z_1, z_2, z\}$ . However  $\beta, \alpha_3, z \in N(z_1) \cap N(z_2)$ , a contradiction. Hence  $R \subseteq N(\beta)$  and  $d(\beta) \geq |R|$ . Therefore,

$$2e(G) = \sum \{d(x) : x \in V\} \geq 3|R| + d(\beta) + 4(n - |R| - 1) \geq 4n - 4. \quad \square$$

**Lemma 6.2.** *If  $G$  is a  $\text{sat}(n, K_{2,3})$ -graph with  $\delta(G) = 3$ , then  $e(G) \geq 2n - 3$ .*

*Proof.* We choose a degree 3 vertex  $\alpha$  with  $e(N(\alpha)) = \lambda(G)$ . By Lemma 6.1, we assume  $\lambda(G) \leq 2$ . Denote  $N(\alpha) = \{x_1, x_2, x_3\}$ , where  $x_3 \notin N(x_2)$ . We define  $V_0 = N[\alpha]$ ,  $\mathcal{U}_1 = \{y \in V \setminus V_0 : |N(y) \cap N(\alpha)| = 2\}$ ,  $\mathcal{U}_2 = \{y \in V \setminus V_0 : |N(y) \cap N(\alpha)| = 1\}$ ,  $\mathcal{U}_3 = V \setminus (V_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2)$ ,

$$\omega(y) = |N(y) \cap (V_0 \cup \mathcal{U}_1 \cup \mathcal{U}_{i-1})| + 0.5|N(y) \cap \mathcal{U}_i| - 2 \text{ for } y \in \mathcal{U}_i, \text{ and } \theta_i = \sum_{y \in \mathcal{U}_i} \omega(y), \quad i = 2, 3.$$

Hence

$$\begin{aligned} e(G) &= e(V_0) + e([V_0, \mathcal{U}_1]) + e(\mathcal{U}_1) + e([V_0 \cup \mathcal{U}_1, \mathcal{U}_2]) + e(\mathcal{U}_2) + e([V \setminus \mathcal{U}_3, \mathcal{U}_3]) + e(\mathcal{U}_3) \\ &= 3 + \lambda(G) + 2|\mathcal{U}_1| + e(\mathcal{U}_1) + 2|\mathcal{U}_2| + \theta_2 + 2|\mathcal{U}_3| + \theta_3 = 2n - 5 + \lambda(G) + e(\mathcal{U}_1) + \theta_2 + \theta_3. \end{aligned}$$

It is sufficient to prove

$$\lambda(G) + e(\mathcal{U}_1) + \theta_2 + \theta_3 \geq 1.5. \quad (16)$$

**Claim 6.2.1.** If  $\lambda(G) = 0$ , then (16) holds.

*Proof.* Since  $e(N(\alpha)) = 0$ , by Corollary 3.3, if  $y \in \mathcal{U}_2 \cup \mathcal{U}_3$ , then  $\omega(y) \geq 0$ . Let  $W_k = \{y \in \mathcal{U}_2 \cup \mathcal{U}_3 : \omega(y) \geq 0.5k\}$ . Note that  $W_3 \subseteq W_2 \subseteq W_1$  and  $\theta_2 + \theta_3 \geq 0.5(|W_1| + |W_2| + |W_3|)$ . First let  $\mathcal{U}_1 = \emptyset$ . Let there exist  $b_i \in \mathcal{U}_2 \cap N(x_i)$  adjacent to  $z_i \in \mathcal{U}_3$  or having  $\omega(b_i) > 0$  for  $i = 1, 2$ , and  $3$ . Since  $\delta(G) = 3$ , by Corollary 3.3,  $z_i \in W_1$ , and if  $z_i = z_j$  and  $i \neq j$  then  $z_i \in W_2$ . Also, if  $z_1 \in N(b_1) \cap N(b_2) \cap N(b_3)$ , then  $z_1 \in W_3$ . Thus  $|W_1| + |W_2| + |W_3| \geq 3$ . Hence we assume that if  $y \in N(x_1)$ , then  $\omega(y) = 0$  and  $N(y) \subseteq V \setminus \mathcal{U}_3$ . Since  $\delta(G) = 3$ , there are  $y, y' \in N(x_1) \cap \mathcal{U}_2$ . By Corollary 3.2, there are  $y_1, y_2 \in N(x_i)$  such that  $N(y) = \{x_1, y_1, y_2\}$ . Without loss of generality,  $i \in \{1, 2\}$ . Since  $N(x_1) \cap N(x_3) = \{\alpha\}$  and  $N(y_1) \cap N(y_2) = \{x_i, y\}$ , it follows  $|N(b) \cap N(y)| \leq 1$  for all  $b \in N(x_3)$ . Since  $yx_3 \notin E(G)$ , by Proposition 3.1,  $y_1$  or  $y_2$ , say  $y_1$ , is adjacent to  $z_1, z_2 \in N(x_3)$ . Thus  $\omega(y_1) \geq 0.5$ . By our assumption,  $i = 2$ . Since  $N(x_1) \cap N(x_2) = \{\alpha\}$ ,  $N(y_1) \cap N(y_2) = \{x_2, y\}$ , and  $yx_2 \notin E(G)$ , by Proposition 3.1,  $y_j$  is adjacent to  $v_1, v_2 \in N(x_2)$ ,  $j = 1$  or  $2$ . If  $j = 1$ , then  $y_1 \in W_3$ . Thus  $j = 2$  and  $y_1, y_2 \in W_1$ . If  $y_k$  is adjacent to  $y'$ ,  $k = 1$  or  $2$ , then  $y_k \in W_2$  and  $|W_1| + |W_2| \geq 3$ . Let  $|\{y_1, y_2\} \cap N(y')| = 0$ . Similarly to  $y$ , the vertex  $y'$  has a neighbor  $y_3 \in W_1$ . Thus  $|W_1| \geq 3$ . Hence we assume  $\mathcal{U}_1 \neq \emptyset$ .

Let  $u_1 \in N(x_1) \cap N(x_2)$  and let  $|N(u_1) \cap \mathcal{U}_1| = 0$ . By Corollary 3.2,  $u_1$  is adjacent to  $y_1, y_2 \in N(x_i) \cap \mathcal{U}_2$ , where  $i \in \{1, 3\}$ . Let  $z_j \in N(y_j) \setminus \{x_i, u_1\}$ ,  $j = 1, 2$ . By Corollary 3.3,  $y_j \in W_2$ ,  $y_j \in W_1$ , or  $z_j \in W_1$  when  $z_j$  is in  $\mathcal{U}_1, \mathcal{U}_2$ , or  $\mathcal{U}_3$ , respectively. Since  $N(y_1) \cap N(y_2) = \{x_i, u_1\}$ , it follows  $z_j \notin N(y_{3-j})$  and  $|W_1 \cap \{y_1, y_2, z_1, z_2\}| \geq 2$ . If  $e(\mathcal{U}_1) > 0$ , or  $z_j \in \mathcal{U}_1$ , or there is  $z \in N(y_j) \setminus \{u_1, x_i, z_j\}$ , then (16) holds. Thus  $e(\mathcal{U}_1) = 0$  and  $N(y_j) = \{u_1, x_i, z_j\}$ , where  $z_j \notin \mathcal{U}_1$ ,  $j = 1, 2$ . Let there be  $u_2 \in \mathcal{U}_1 \setminus \{u_1\}$ . By Corollary 3.3, there are  $y_3, y_4 \in N(x_k) \cap N(u_2)$ . Similarly to  $y_1, y_2$ ,  $N(y_3) = \{u_2, x_k, z_3\}$  and  $z_3 \notin \mathcal{U}_1$ . If  $z_3 \in \{y_1, y_2, z_1, z_2\} \cap \mathcal{U}_2$  or  $z_3 \notin \{z_1, z_2\}$ , then  $|W_1| \geq 3$ . Thus  $z_1 = z_3$  and  $z_1 \in \mathcal{U}_3$ . Since  $y_1 y_3 \notin E(G)$  and  $z_1 \in \mathcal{U}_3$ , by Proposition 3.1,  $z_1 \in (N(u_1) \cup N(u_2)) \cap W_2$  and (16) holds. Hence  $\mathcal{U}_1 = \{u_1\}$ .

Since  $d(x_2) \geq 3$ , there is  $y_5 \in N(x_2) \cap \mathcal{U}_2$ . Let  $y_5 \in N(u_1)$  and  $z_5 \in N(y_5) \setminus \{u_1, x_2\}$ . If  $y_5 = z_j$ ,  $j = 1$  or  $2$ , then  $y_j, z_j \in W_1$  and  $|W_1| \geq 3$ . Since  $i \neq 2$ , we assume  $y_5 \notin \{y_1, y_2, z_1, z_2\}$ . By Corollary 3.3,  $y_5 \in W_1$  or  $z_5 \in W_1$  when  $z_5$  is in  $\mathcal{U}_2$  or  $\mathcal{U}_3$ , respectively. Also, if  $z_5 \in \{z_1, z_2\}$ , then  $z_5 \in W_2$ . Thus  $|W_1| + |W_2| \geq 3$ . Hence  $y_5 \notin N(u_1)$ . By Corollary 3.2, there are  $y_6, y_7 \in N(x_k) \cap N(y_5)$ . If there is  $z \in N(y_5) \setminus \{x_2, y_6, y_7\}$ , then by Corollary 3.3,  $\{y_5, z\} \cap W_1 \neq \emptyset$ . Also, if  $z \in \{z_1, z_2\}$ , then  $z \in W_2$ . Hence  $N(y_5) = \{x_2, y_6, y_7\}$ . Since  $N(x_3) \cap N(x_2) = \{\alpha\}$ ,  $N(y_6) \cap N(y_7) = \{x_k, y_5\}$ ,  $x_k \notin N(x_3)$ , and  $y_5 x_3 \notin E(G)$ , by Proposition 3.1,  $y_6$  or  $y_7$ , say  $y_6$ , is adjacent to  $w_3, w_4 \in N(x_3)$ . Thus  $y_6 \in W_1$ . Since  $N(y_j) = \{u_1, x_i, z_j\}$  and  $d(y_6) \geq 4$ , it follows  $y_6 \neq y_j$ ,  $j = 1, 2$ . If  $y_6 = z_j$ ,  $j = 1$  or  $2$ , then  $y_j \in W_1$ . Thus  $|W_1| + |W_2| \geq 3$ . Hence  $|N(u) \cap \mathcal{U}_1| \geq 1$  if  $u \in \mathcal{U}_1$ .

We can assume  $E(G[\mathcal{U}_1]) = \{u_1 u_2\}$ ,  $|\mathcal{U}_1| = 2$ , and  $u_2 \in N(x_2) \cap N(x_3)$ . Since  $d(x_1) \geq 3$ , there is  $y^* \in N(x_1) \cap \mathcal{U}_2$ . If there is  $z \in (\mathcal{U}_3 \cup \mathcal{U}_1) \cap N(y^*)$ , then since  $\delta(G) = 3$ , by Corollary 3.3,  $(\{y^*\} \cup (N(y^*) \cap \mathcal{U}_3)) \cap W_1 \neq \emptyset$ . Let  $N(y^*) \subseteq \{x_1\} \cup \mathcal{U}_2$ . If there is  $v \in N(y^*)$  adjacent to distinct  $v_1, v_2 \in N(x_3)$ , then  $v \in \mathcal{U}_2 \cap W_1$ . Let  $|N(v) \cap N(x_3)| \leq 1$  for all  $v \in N(y^*)$ . Since  $y^* x_3 \notin E(G)$  and  $N(x_3) \cap N(x_1) = \{\alpha\}$ , by Proposition 3.1, there is  $z \in N(x_3)$  adjacent to  $z_1, z_2 \in N(y^*) \setminus \{x_1\}$ . By Corollary 3.2,  $z_1 \in W_1$  and (16) holds.  $\square$

From now on, we assume  $\lambda(G) = 1$  or  $2$  and  $x_1 \in N(x_2)$ . Recall  $x_3 \notin N(x_2)$ . We define

$$\mathcal{U}_2^+ = \{y \in \mathcal{U}_2 : \omega(y) \geq 0.5\}, \mathcal{U}_2^- = \{y \in \mathcal{U}_2 : \omega(y) < 0\}, \text{ and } \mathcal{U}_{2i}^- = \mathcal{U}_2^- \cap N(x_i), i = 1, 2, 3,$$

$$\mathcal{U}_3^g = \{z \in \mathcal{U}_3 : d(z) \geq 4 \text{ or } |N(z) \cap (V \setminus \mathcal{U}_3)| \geq 3 \text{ or } N(z) \cap (\mathcal{U}_1 \cup (\mathcal{U}_2 \setminus \mathcal{U}_2^-)) \neq \emptyset\},$$

and  $\mathcal{U}_3^b = \mathcal{U}_3 \setminus \mathcal{U}_3^g$ . For  $y \in \mathcal{U}_2^-$ , we define  $f(y)$ , a subset of  $N(y) \cap \mathcal{U}_3$  satisfying (17):

$$\text{If } z \in f(y) \text{ and } z \notin \mathcal{U}_3^g, \text{ then there is } y_1 \in \mathcal{U}_2^- \cap N(z) \text{ with } z \notin f(y_1). \quad (17)$$

Let  $y \in \mathcal{U}_{2j}^-$  and  $\omega(y) = -0.5$ . By Corollary 3.3, there is  $y' \in N(x_i)$  such that  $N(y) \cap (V \setminus \mathcal{U}_3) = N(y) \cap N(x_i) = \{x_j, y'\}$ . Since  $d(y) \geq 3$ , there is  $z \in N(y) \cap \mathcal{U}_3$ . We choose one  $z \in N(y) \cap \mathcal{U}_3^g$  and define  $f(y) = \{z\}$ , if  $N(y) \cap \mathcal{U}_3^g \neq \emptyset$ . Let  $N(y) \cap \mathcal{U}_3^g = \emptyset$  and  $z \in N(y) \cap \mathcal{U}_3$ . By Corollary 3.2, there is  $y_1 \in \mathcal{U}_{2j}^-$  and  $w \in \mathcal{U}_3$  such that  $N(z) = \{y, y_1, w\}$ . By Corollary 3.3,  $yy_1 \notin E(G)$ . Since  $N(y) \cap N(y_1) = \{x_j, z\}$  and  $\lambda(G) \geq 1$ , it follows  $w$  is adjacent to exactly one of  $y, y_1$ ,  $\lambda(G) = 1$ ,  $\{i, j\} = \{1, 2\}$ , and  $x_3 \notin N(x_j)$ . Since  $N(y) \cap N(y_1) = \{x_j, z\}$ ,  $N(b) \cap N(z) \subseteq \{w\}$  for all  $b \in N(x_3)$ , and  $zx_3 \notin E(G)$ , by Proposition 3.1,  $w$  is adjacent to  $y_2, y_3 \in N(x_3)$ . Thus  $w \in \mathcal{U}_3^g \cap N(y_1)$  and  $z \notin f(y_1)$ . We define  $f(y) = \{z\}$  and  $f(y)$  satisfies (17).

Next let  $y \in \mathcal{U}_2^-$  with  $\omega(y) = -1$ . By Corollary 3.3,  $y \in N(x_1)$ ,  $x_1 \in N(x_2) \cap N(x_3)$ , and  $\lambda(G) = 2$ . Since  $d(y) \geq 3$  and  $N(y) \cap (V \setminus \mathcal{U}_3) = \{x_1\}$ , there are  $z, z' \in N(y) \cap \mathcal{U}_3$ . We claim

$z, z' \in \mathcal{U}_3^g$ . Let  $z \in \mathcal{U}_3^b$  and  $N(z) = \{y, y_1, w\}$ , where  $y_1 \in \mathcal{U}_2^-$  and  $w \in \mathcal{U}_3$ . By Corollary 3.2,  $y_1 \in N(x_1)$  and  $yy_1 \notin E(G)$ . Since  $\lambda(G) = 2$ , it follows  $x_1, z, w \in N(y) \cap N(y_1)$ , a contradiction justifying our claim. We define  $f(y) = \{z, z'\}$  and  $f(y) \subseteq \mathcal{U}_3^g$  satisfying (17).

We claim that

$$\text{if } z \in \mathcal{U}_3, \text{ then } \omega(z) \geq \epsilon + 0.5|\{y \in \mathcal{U}_2^- : z \in f(y)\}|, \quad (18)$$

where  $\epsilon = 0.5$  if  $x_3 \notin N(x_1)$  and  $|N(z) \cap N(x_3) \cap \mathcal{U}_2| \geq 1$  or  $|N(z) \cap N(x_3) \cap \mathcal{U}_1| \geq 2$ , and  $\epsilon = 0$ , otherwise.

Let  $z \in \mathcal{U}_3$ . By Corollary 3.2,  $N(z) \cap (\mathcal{U}_1 \cup \mathcal{U}_2) \neq \emptyset$ . Also, if  $x_3 \notin N(x_1)$ , then  $N(x_3) \cap \mathcal{U}_2 \subseteq \mathcal{U}_2^+$ , since  $x_3 \notin N(x_2)$ . Since  $d(z) \geq 3$ , (18) follows Corollary 3.3 if  $z \notin f(y)$  for any  $y$ . Let  $z \in f(y)$ , where  $y \in \mathcal{U}_2^-$ . By Corollary 3.3,  $|N(z) \cap (V \setminus \mathcal{U}_3)| \geq 2$ . First let  $N(z) \cap (V \setminus \mathcal{U}_3) = \{y, y_1\}$ . Since  $d(z) \geq 3$  and  $f(y)$  satisfies (17), it follows  $z \in \mathcal{U}_3^g$  or  $z \notin f(y_1)$ , and (18) holds. Let  $|N(z) \cap (V \setminus \mathcal{U}_3)| = 3$ . Since  $\omega(z) \geq 1 + 0.5(d(z) - 3)$ , (18) holds if  $d(z) \geq 4$  or  $|N(z) \cap \mathcal{U}_1| > 0$ . Let  $d(z) = 3$  and  $N(z) \subseteq \mathcal{U}_2$ . If  $\lambda(G) = 2$ , then  $\epsilon = 0$ , there is  $y' \in N(z) \setminus \mathcal{U}_2^-$  with  $|N(y') \cap N(z)| = 2$ , and (18) holds. Let  $E(G[N(\alpha)]) = \{x_1x_2\}$  and  $y_1y_2 \in E(G[N(z)])$ . If  $y_i \notin \mathcal{U}_2^-$ ,  $i = 1$  or  $2$ , then (18) holds. By Corollary 3.3,  $y_i \in N(x_i) \cap \mathcal{U}_2^-$ ,  $i = 1, 2$ . Let  $y_3 \in N(z) \setminus \{y_1, y_2\}$ . Since  $z \notin N(\alpha)$ , by Corollary 3.2,  $y_3 \in N(x_1) \cup N(x_2)$ . Since  $zx_3 \notin E(G)$ , by Proposition 3.1, there is  $b \in N(x_3) \cap N(y_3)$ . By Corollary 3.3,  $y_3 \notin \mathcal{U}_2^-$  and (18) holds. If  $|N(z) \cap (V \setminus \mathcal{U}_3)| \geq 4$ , then  $\omega(z) \geq 0.5|N(z) \cap (V \setminus \mathcal{U}_3)|$  and (18) holds.

By our construction, if  $\omega(y) = -0.5$ , then  $|f(y)| = 1$ . Also, if  $\omega(y) = -1$ , then  $|f(y)| = 2$ . Let  $\ell^* = \sum\{\omega(y) : y \in \mathcal{U}_2^+\}$ . By (18),

$$\theta_2 + \theta_3 = \sum\{\omega(y) : y \in \mathcal{U}_2^+ \cup \mathcal{U}_3 \cup \mathcal{U}_2^-\} \geq \ell^* + \epsilon + 0.5 \sum_{y \in \mathcal{U}_2^-} |f(y)| + \sum_{y \in \mathcal{U}_2^-} \omega(y) \geq \ell^* + \epsilon, \quad (19)$$

where  $\epsilon = 0.5$  if  $x_3 \notin N(x_1)$  and there is  $z \in \mathcal{U}_3$  with  $|N(z) \cap N(x_3) \cap \mathcal{U}_2| \geq 1$  or  $|N(z) \cap N(x_3) \cap \mathcal{U}_1| \geq 2$ , and  $\epsilon = 0$ , otherwise.

By (19), if  $\lambda(G) = 2$  or  $e(\mathcal{U}_1) \geq 1$ , then (16) holds. Let  $\lambda(G) = 1$ ,  $E(G[N(\alpha)]) = \{x_1x_2\}$ , and  $e(\mathcal{U}_1) = 0$ . If there is  $y \in N(x_3) \cap \mathcal{U}_2$  adjacent to a vertex in  $\mathcal{U}_3$  or with  $\omega(y) \geq 0.5$ , then by (19), (16) holds. Thus we assume  $\omega(y) = |N(y) \cap \mathcal{U}_3| = 0$  for  $y \in N(x_3) \cap \mathcal{U}_2$ . Since  $\delta(G) = 3$ ,  $e(\mathcal{U}_1) = 0$ , and  $\lambda(G) = 1$ , there is  $y^* \in N(x_3) \cap \mathcal{U}_2$ . By Corollary 3.2, there are  $x_j \in N(\alpha)$  and  $y_1, y_2 \in N(x_j) \setminus \mathcal{U}_1$  such that  $N(y^*) = \{x_3, y_1, y_2\}$ . Let  $j \neq 3$ . Since  $\lambda(G) = 1$ ,  $y_1y_2 \in E(G)$ . By Corollary 3.2, either  $N(y_1) \cap \mathcal{U}_1 \neq \emptyset$  or there is  $x_i \in N(\alpha)$  with  $|N(y_1) \cap N(x_i)| \geq 2$ . Thus  $\ell^* \geq \omega(y_1) \geq 0.5$  and (16) holds. Hence  $y_1, y_2 \in N(x_3)$  and similarly for  $y_i$ ,  $N(y_i) \subseteq \{x_3\} \cup (N(x_3) \cap \mathcal{U}_2)$ ,  $i = 1, 2$ . First let  $|\mathcal{U}_3| = 0$ . Since  $N(b) \cap N(y^*) \subseteq \{x_3\}$  for all  $b \in N(x_1)$  and  $y^*x_1 \notin E(G)$ , by Proposition 3.1, there is  $z \in N(y^*)$  with  $|N(z) \cap N(x_1)| \geq 2$ . Since  $N(y_i) \subseteq \{x_3\} \cup (N(x_3) \cap \mathcal{U}_2)$ ,  $i = 1, 2$ , it follows  $z = x_3$ . There is  $u \in (N(x_3) \cap N(x_1)) \setminus \{\alpha\}$  and  $u \in \mathcal{U}_1$ . Since  $\delta(G) = 3$  and  $|\mathcal{U}_3| = e(\mathcal{U}_1) = 0$ , there is  $y \in N(u) \cap \mathcal{U}_2$ ,  $\ell^* \geq \omega(y) \geq 0.5$ , and (16) holds. Hence there is  $z^* \in \mathcal{U}_3$ . Since  $N(b) \cap N(y^*) \subseteq \{x_3\}$  for all  $b \in N(z^*)$  and  $y^*z^* \notin E(G)$ , by Proposition 3.1, there is  $w \in N(y^*)$  with  $|N(w) \cap N(z^*)| \geq 2$ . Since  $N(y_i) \subseteq \{x_3\} \cup (N(x_3) \cap \mathcal{U}_2)$ ,  $i = 1, 2$ , and  $|N(y) \cap \mathcal{U}_3| = 0$  for all  $y \in N(x_3) \cap \mathcal{U}_2$ , it follows  $w = x_3$ ,  $|N(x_3) \cap \mathcal{U}_1 \cap N(z^*)| \geq 2$ , and  $\epsilon = 0.5$ . By (19), (16) holds justifying Lemma 6.2.  $\square$

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